Complete Constraint Satisfaction Problems

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CSP

CSP = Communicating Sequential Processes

CSP

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CSP = constraint satisfaction problem

CSP examples

- allocating frequencies to mobile phone cells
- checking if a logical formula is satisfiable
- laying out components on circuit board
- fitting a protein structure to measurements
- finding DNA sequence from set of contigs
- drawing up a timetable
- solving system of linear equations

Constraint satisfaction

assign values to variables to satisfy constraints

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What is a constraint?

each constraint has two parts:

- scope: ordered list of variables
- relation: allowed combinations of values
- relation R of arity $\rho(R)$ over a domain D:
  $R \subseteq D^{\rho(R)}$

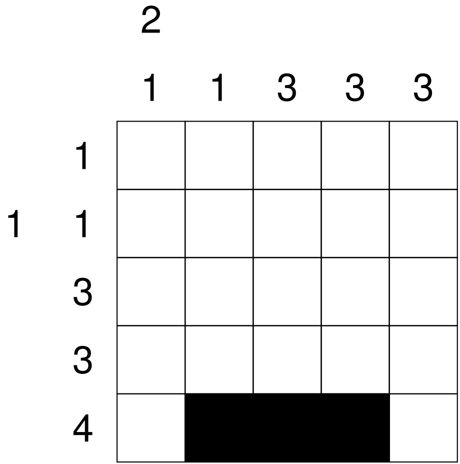
Nonogram

		2				
		1	1	3	3	3
	1					
1	1					
	3					
	3					
	4					

Nonogram

		2					
		1	1	3	3	3	
	1						
1	1						← easy
	3						
	3						
	4						

Nonogram



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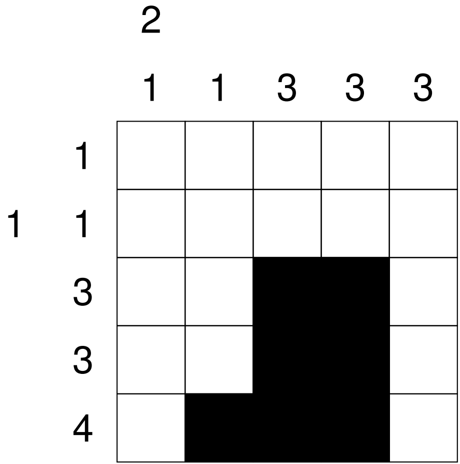

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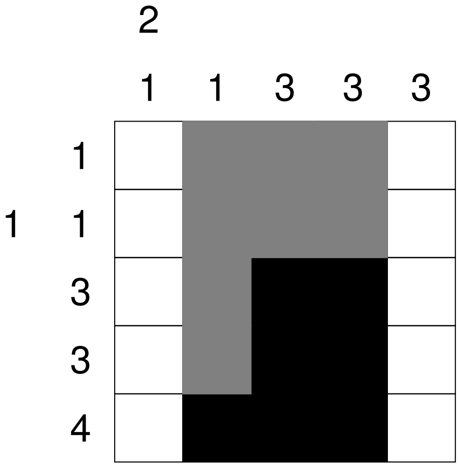

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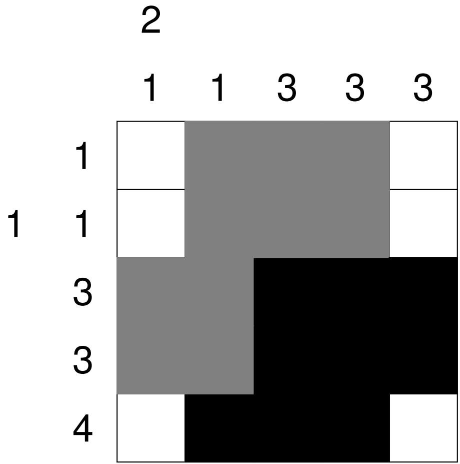

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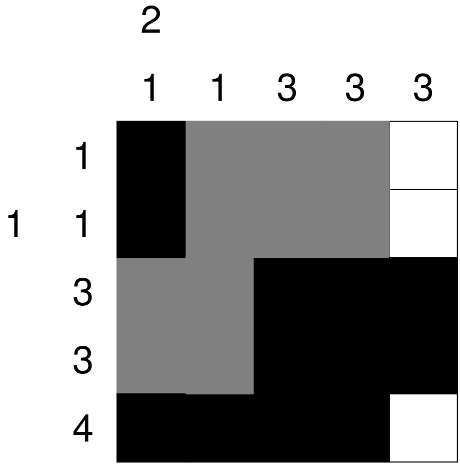

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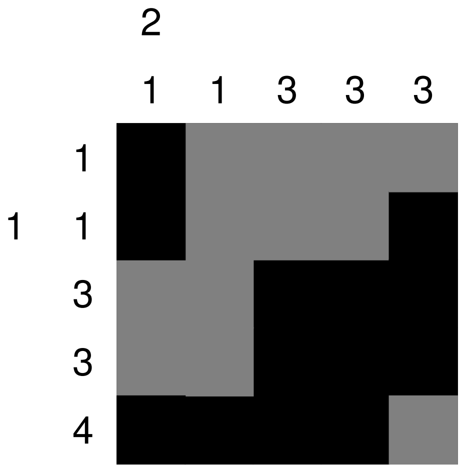
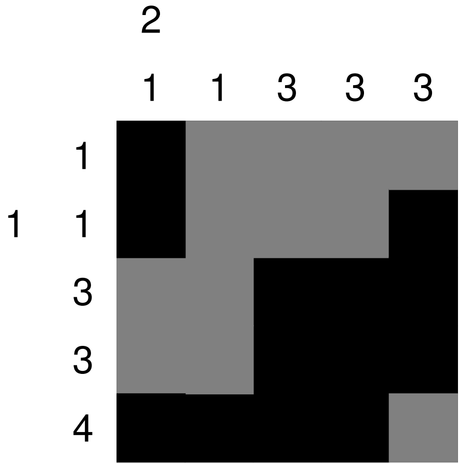


Nonogram (CSP)



Variable-value representation

Constraint satisfaction problem instance

- variables: set V of size $|V| = s$
- values: set D of size $|D| = t$
- constraints: set C
- solution: function $f : V \to D$ such that if $((v_1, v_2, \ldots, v_r), R) \in C$ then $(f(v_1), f(v_2), \ldots, f(v_r)) \in R$

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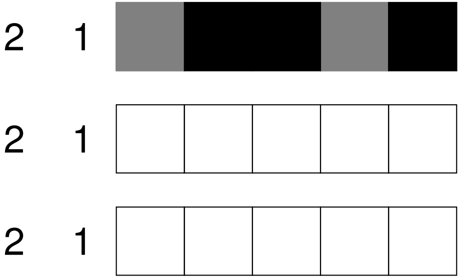
instance: (V, D, C)

Nonogram constraint

possible patterns



Nonogram constraint

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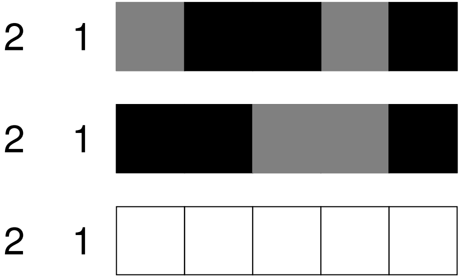

Nonogram constraint

possible patterns

Nonogram constraint

possible assignments

v_{00} v_{01} v_{02} v_{03} v_{04} 0 1 1 0 1

Nonogram constraint

possible assignments

					0	1	1	0	1
v_{00}	v_{01}	v_{02}	v_{03}	v_{04}	1	1	0	0	1

Nonogram constraint

possible assignments

					0	1	1	0	1
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v_{00}	v_{01}	v_{02}	v_{03}	v_{04}	1	1	0	1	0

Nonogram constraint

$$\left(\begin{array}{ll} (v_{00}, v_{01}, v_{02}, v_{03}, v_{04}), \; \{ & (0,1,1,0,1), \\ & (1,1,0,0,1), \\ & (1,1,0,1,0) \; \} \end{array} \right)$$

All-different constraint

all-different constraint of arity r over domain D

- scope: variables $(v_1, v_2, \ldots, v_r)$
- relation: all tuples $(d_1, d_2, \ldots, d_r)$ over D such that $i \neq j \Rightarrow d_i \neq d_j$

all-different relation is denoted $\neq_D^r$ or just $\neq$

Sudoku

4	6				1			
		2		9	6			
	3						6	8
							3	5
			6		5			
7	1							
8	4						7	
			5	1		9		
			3				2	4

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← hard

Relational structure

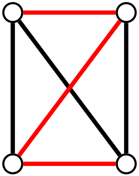
tuple $(D, (R_i)_{i \in I})$

- domain D
- tuple $(R_i)_{i \in I}$ of relations
- ordered index set I
- relations R_i over D: $R_i \subseteq D^{\rho(R_i)}$

Graph as relational structure

complemented representation: $(V, (E_i)_{i \in \{1,2\}})$

- E_1 symmetric irreflexive relation over V (arity 2)
- E_2 complement of E_1
- colour E_1 edges black, E_2 red



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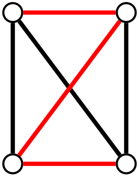
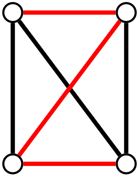


Graph as relational structure

complemented representation: $(V, (E, \overline{E}))$

- E symmetric irreflexive relation over V (arity 2)
- $\overline{E}$ complement of E
- colour E edges black, $\overline{E}$ red



Homomorphism representation

Constraint satisfaction problem instance

- source: $S = (V, (Q_i)_{i \in I})$, $|V| = s$
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- S and T similar: arities of Q_i and R_i match
- solution: function $f : V \to D$ s.t. $\forall i \in I$

$$(v_1, \ldots, v_{\rho(R)}) \in Q_i \Rightarrow (f(v_1), \ldots, f(v_{\rho(R)})) \in R_i$$

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instance: $S \to T?$

Graph isomorphism

Are graphs $G = (V, Q)$, $H = (D, R)$ isomorphic?

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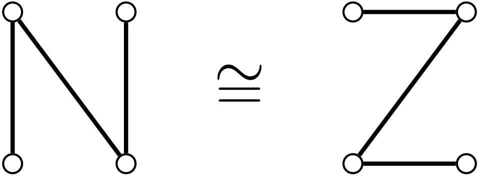
G is isomorphic to H if and only if $S \to T$



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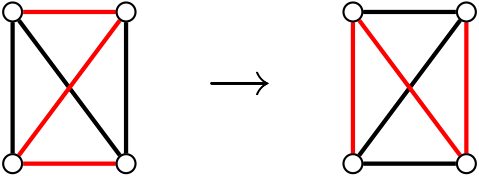
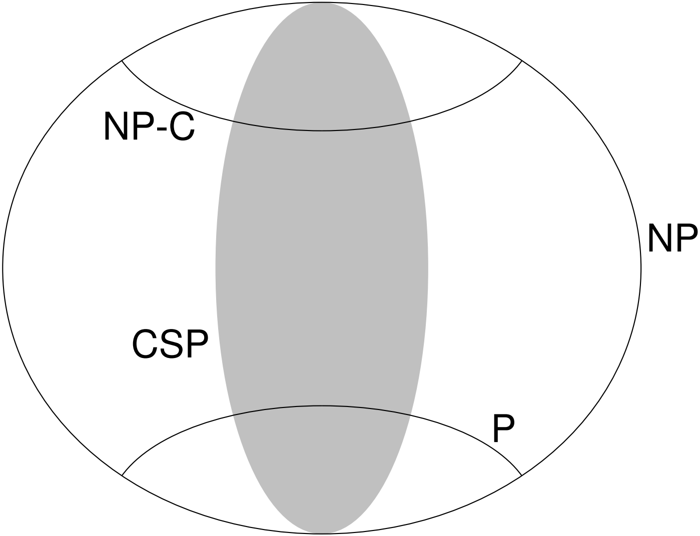


Constraint satisfaction problems

- problem: class of instances
- CSP($\mathcal{S}$, _) = $\{S \to \_ \mid S \in \mathcal{S}\}$
- captures SUBGRAPH ISOMORPHISM, CLIQUE, GRAPH ISOMORPHISM, ...
- CSP(_, $\mathcal{T}$) = $\{\_ \to T \mid T \in \mathcal{T}\}$
- captures SATISFIABILITY, GRAPH COLOURING, 2-SAT, ...

CSP vs. NP



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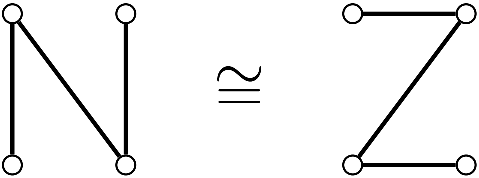
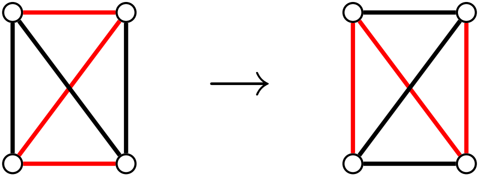
GRAPH ISOMORPHISM

$$(V, Q) \cong (D, R)?$$



GRAPH ISOMORPHISM

$$(V,(Q,\overline{Q})) \to (D,(R,\overline{R}))?$$

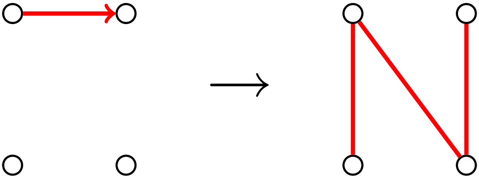


GRAPH ISOMORPHISM

$$CS(2, s) \to (D, (R_i)_{i \in I})?$$

GRAPH ISOMORPHISM

$$(\{0,1,2,3\},(\{(0,1)\},\ldots)_{i\in I}) \rightarrow (D,(R_i)_{i\in I})$$



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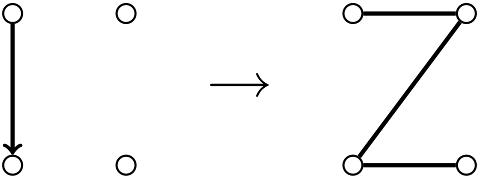

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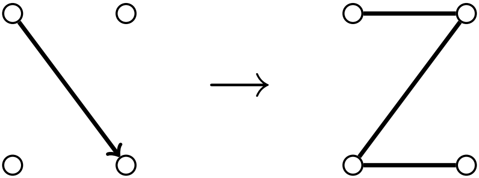

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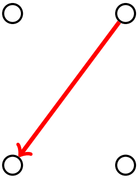
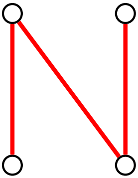


$$\longrightarrow$$



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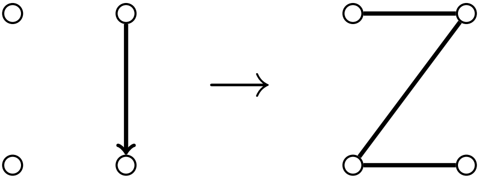
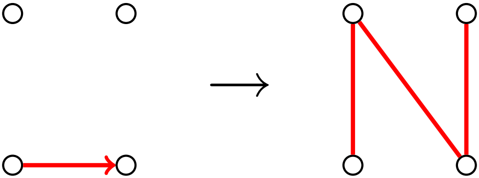


GRAPH ISOMORPHISM

$$(\{0,1,2,3\}, (\ldots, \{(2,3)\})_{i\in I}) \to (D, (R_i)_{i\in I})$$



What is going on?

- source embedded in target
- ...via pattern of relations
- r-dimensional adjacency structure

Why do this?

- ordered so nice data structures (arrays, linked lists, mixed-radix systems)
- CR is often same size: no need to represent relations explicitly
- vary parameters r, s, t: different problems

Complexity

- q is number of unique relations in CR
- instance size $n \sim qt^r$ bits
- exhaustive enumeration $\leq qrt^s$ steps
- upper bound: $\sim n^{s/r}$ steps

Complexity

upper bounds

- $s/r \to \infty$: exponential bound
- $s/r \leq c$: $n^{s/r}$ at most polynomial
- s/r grows, but slowly
  - unlikely NP-complete
  - not known to be tractable
  - (candidate) intermediate problems?

A theorem of Grohe

Theorem: Assume FPT $\neq$ W[1]. Let $\mathcal{S}$ be a recursively enumerable class of structures of bounded arity. Then CSP($\mathcal{S}$,_) is tractable iff $\mathcal{S}$ has bounded tree width modulo homomorphic equivalence. (Grohe 2003)

. . . vs. CR

- tw$(CS(r, s)) = s - 1$
- $CS(r, s) \leftrightarrow T \Rightarrow$ tw$(T) \geq s - 1$
- Let $\mathcal{S} = \{CS(2, s) \mid s \geq 1\}$
- $\mathcal{S}$ has unbounded tree width (even mhe)
- CS: *completely determines* tw (even mhe)

so: FTP $\neq$ W[1] $\Rightarrow$ CSP($\mathcal{S}$, _) is not tractable
(note: FTP $\neq$ W[1] $\Rightarrow$ P $\neq$ NP)

A conjecture of Grohe

Conjecture: For every class $\mathcal{S}$ of structures, CSP($\mathcal{S}$, _) is tractable iff $\mathcal{S}$ has bounded fractional hypertree width modulo homomorphic equivalence. (Grohe 2006)

...vs. CR

- fhw$(CS(r, s)) = s/r$
- $CS(r, s) \leftrightarrow T \Rightarrow$ fhw$(T) \geq s/r$
- Let $\mathcal{S}_x = \{CS(r, s) \mid s/r \sim x\}$
- $\mathcal{S}_x$ has unbounded fhw (even mhe) if e.g. x is inverse Ackermann
- CS: *completely determines* fhw (even mhe)

yet: such problems have near-polynomial upper bounds on runtime

Summary

complete representation

- s/r
- CR fixes tw, fhw (even mhe)
- tool for CSP($\mathcal{S}$, _)

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Languages

- $\Gamma(P)$: set of all relations in target structures
- expressibility: $\langle\Gamma\rangle$ characterises tractability (Jeavons 1996)
- local language: tuples of relations in each instance

Local languages

$CS(2,s) \to T$	$\{R_i\}$	r,s,t
LOG-CLIQUE	E	$t = 2^s$
s-CLIQUE	E	
t-COLOURING	$*, \neq$	
GRAPH ISOMORPHISM	$E, \overline{E}$	$s = t$
SUBGRAPH ISOMORPHISM	$E, \overline{E}$	

$CS(r,s) \to T$		
HYPERGRAPH t-COLOURING	$*, \neq$	
COMPLETE $(r, \_, \_)$-CSP	$R_1, \ldots, R_q$	

E *is a symmetric irreflexive relation*