ACPSS 2008 Modelling Competition: Team PDLP

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2 July 2008

What is problem?

"nontransitive dice"

- 3 dice
- each die has 6 sides
- sides are labelled with positive integers
- labels are not necessarily unique

What is problem actually?

characterize for which K, S, D there exists a probability space such that

- there exist K events $A_1, A_2, \ldots, A_K$
- each event has S equiprobable outcomes
- there exists a labelling of outcomes with integers $1, 2, \ldots, D$
- the $K * S$ labels are not necessarily unique
- $Pr[A_K > A_1] \geq \pi$ and $Pr[A_i > A_{i+1}] \geq \pi$ for each $i = 1, 2, \ldots, K-2$
- $\pi > 1/2$

Problem

characterize for which K, S, D

- there exist K dice $A_1, A_2, \ldots, A_K$
- each die has S equiprobable faces
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Problem

characterize for which K, S, D

- there exist K dice $A_1, A_2, \ldots, A_K$
- each die has S equiprobable faces
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Known

Steinhaus & Trybuła 1959, 1961; Trybuła 1965

1. there exists a sample space with events X, Y, and Z, such that $Pr[X > Y] > 0.5$, $Pr[Y > Z] > 0.5$, and $Pr[Z > X] > 0.5$;
2. Suppose $D = S * K$, and all labels are distinct. Then

$$\bigvee_{i=0}^{K-1} (Pr[A_i > A_{i+1}] < p_{K-1}).$$

Here $p_2 = \Phi \approx 0.618$, $p_3 = 2/3$, $\lim_{n\to\infty} p_n = 3/4$.

Prototype

- Gecode implementation
- used for some validation

Model

- variable $A[i,j]$ is label of j-th face of die i
- variable $P[i]$ is probability numerator of pair $(i, i+1)$
- constraints

```
forall i: int( 0..(K-1) )
  . ( sum j,k: int( 0..(S-1) )
          . ( A[i,j]>A[((i+1)%K),k] )
    ) >= P[i]
```

Symmetry

Cohen, Jeavons, Jefferson, Petrie, Smith 2006

- constraint symmetry is automorphism of microstructure
- solution symmetry of k-ary instance is automorphism of k-ary nogood hypergraph

Shift

- change solution: $2, 8, 9$ / $5, 6, 7$ / $3, 4, 10$
- to solution: $1, 7, 8$ / $4, 5, 6$ / $2, 3, 9$

Remove symmetry

- set of dice is represented as array
- need to remove spurious duplicates
- also remove spuriously shifted values
- `A[0,0]=1`

Remove more symmetry

- set of face labels is represented as array
- need to remove spurious duplicates
- ```
  forall i: int( 0..(K-1) )
  . (forall j: int( 0..(S-2) )
     . (A[i,j] <= A[i,j+1]))
  ```
  ```

Remove yet more symmetry

Tenney & Foster 1976

- advantage table
- number of faces dominated by any face
-
```
forall i: int( 0..(K-1) )
. ( forall j: int( 0..(S-1) )
    . ( sum k: int( 0..(S-1) )
        . ( A[i,j]>A[((i+1)%K),k] ) )
      <= S
  )
```

Remove nothing

Tenney & Foster 1976

- advantage table
- number of faces dominated by any face
-
```
forall i: int( 0..(K-1) )
. ( forall j: int( 0..(S-1) )
    . ( sum k: int( 0..(S-1) )
        . ( A[i,j]>A[((i+1)%K),k] ) )
      <= S
  )
```
- note: $k \leq S \Rightarrow \sum_k \delta_{A[i,j]>A[i+1,k]} \leq S$

$K = 3$, $P[i] = P[j]$

1+9i	5+9i	9+9i	...
3+9i	4+9i	8+9i	...
2+9i	6+9i	7+9i	...

S can be any multiple of 3

Different probability

Our basic approach.

Largest set of dice

Unbounded, see Savage 1994 for algorithm.

Other questions

- What is minimum number of sides with given number of dice?
- What is maximum probability that one die wins against another?

C1 find configuration with maximal $\min_{i=0}^{K-1} P[i]$

C2 find configuration with maximal $\sum_{i=0}^{K-1} P[i]$

- hypothesis: distinct vs. non-distinct matters

Varying S

- no solutions for $S < 3$
- start with $S = 3$
- solution: $1, 7, 8$ / $4, 5, 6$ / $2, 3, 9$
- …lots of others
- (this one has largest excess probability)

Efron's dice

1	1	5	5	5	5
4	4	4	4	4	4
3	3	3	3	7	7
2	2	2	6	6	6

unique solution with $K = 4, S = 6, D = 7$
also found for $K = 4, S = 6, D = 9$ via C1: 387K nodes, 3 s
also found for $K = 4, S = 6, D = 9$ via C2: 18m nodes, 3 min
lots of experiments, data remains to be analysed...

Conclusion

- general case, reasonable model
- Found 1 verified Minion bug, 2 verified Tailor bugs

Future work

- improve: remove reliance on reification
- reformulate: use advantage table as variables
- change: investigate Condorcet voting paradox