Multicoloured cliques in vertex-coloured graphs

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A ***vertex-colouring*** of graph G is a function $c\colon V(G) \to [k(c)]$. A ***vc-graph*** is a pair (G, c).

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$[i] = \{1, 2, \ldots, i\}$.

Can take $k(c) = |c(V(G))|$.

$S \subseteq V(G)$ is ***rainbow*** for colouring $c\colon V(G) \to [k(c)]$ if $|c(S)| = |S|$, ***strong rainbow*** if further $|S| = k(c)$.

IS	
Input:	graph G, integer k
Question:	does G contain independent set of size k?

IS($\mathcal{C}$)	
Input:	graph G, integer k , $G \in \mathcal{C}$
Question:	does G contain independent set of size k?

ISGCI: 1031 classes at present

Theorem (Chen, Huang, Kanj, Xia, 2006): if IS can be solved in $f(k)m^{o(k)}$ time then it can be solved in $2^{o(n)}$ time.

SRIS

Input: vc-graph (G, c)

Question: does (G, c) contain a strong rainbow independent set?

SRIS

Input: vc-graph (G, c)

Question: does (G, c) contain a rainbow independent set of size $|c(V(G))|$?

RIS	
Input:	vc-graph (G, c), integer k
Question:	does (G, c) contain a rainbow independent set of size k?

Multicoloured cliques in vertex-coloured graphs

Multicoloured independent sets in vertex-coloured graphs

Strong rainbow independent sets in vertex-coloured graphs

Suppose $S = (V, (Q_i)_I)$, $T = (W, (R_i)_I)$ are relational structures with $(r(Q_i)_I) = (r(R_i)_I)$.

f is ***homomorphism*** from S to T if $f : V \to W$ such that $(u_1, u_2, \ldots, u_{r(Q_i)}) \in Q_i \Rightarrow$ $(f(u_1), f(u_2), \ldots, f(u_{r(Q_i)})) \in R_i$.

$S \to T \equiv \exists$ homomorphism from S to T

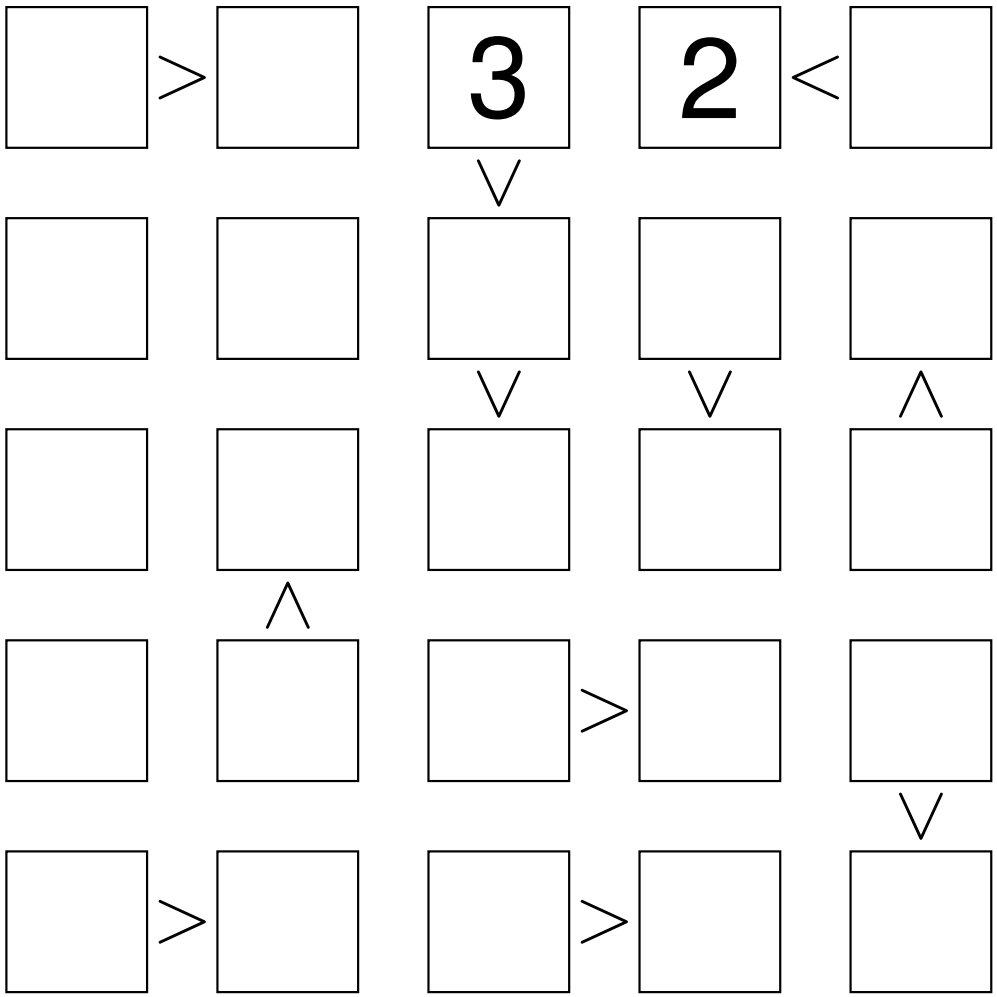
CSP_2	
Input:	arity ≤ 2 relational structures (S, T)
Question:	$S \longrightarrow T$?

$\mathrm{CSP}_2(\mathcal{C},\mathcal{D})$

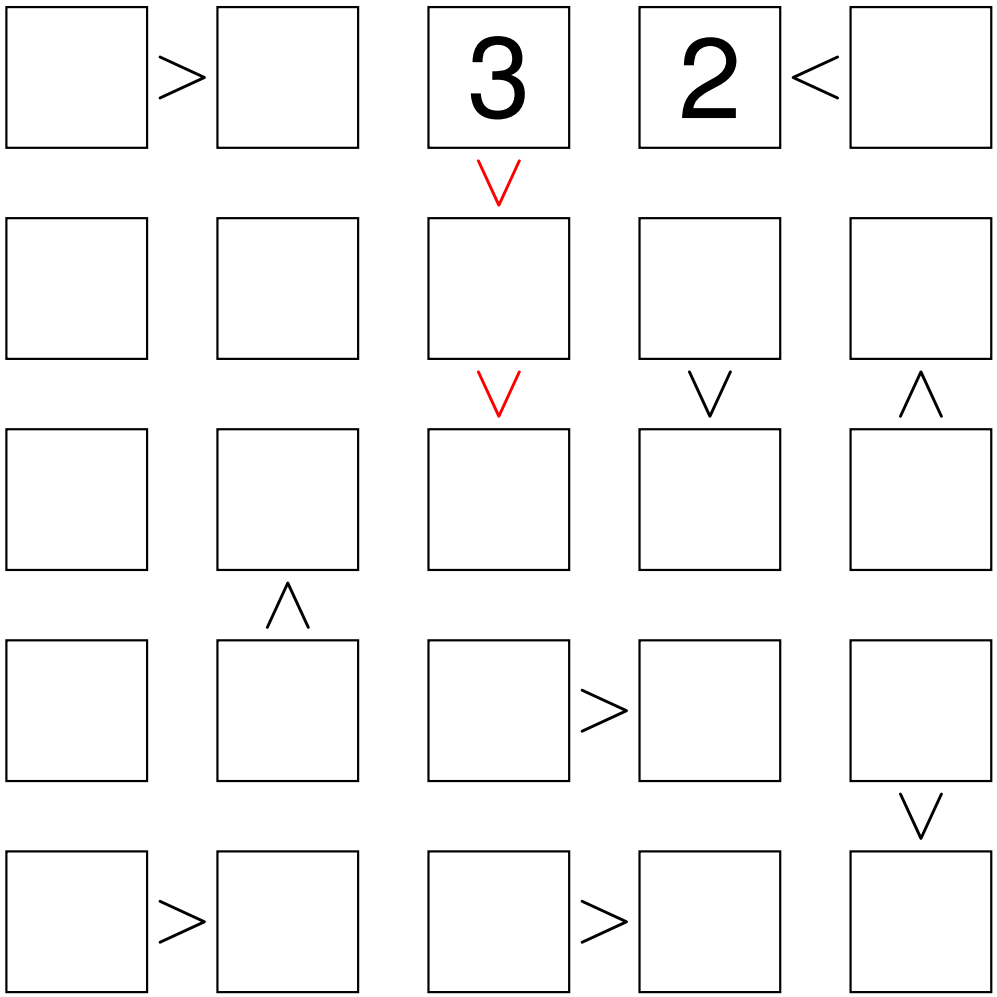
Input: arity $\leq$ 2 relational structures (S,T), $S \in \mathcal{C}, T \in \mathcal{D}$

Question: $S \longrightarrow T$?

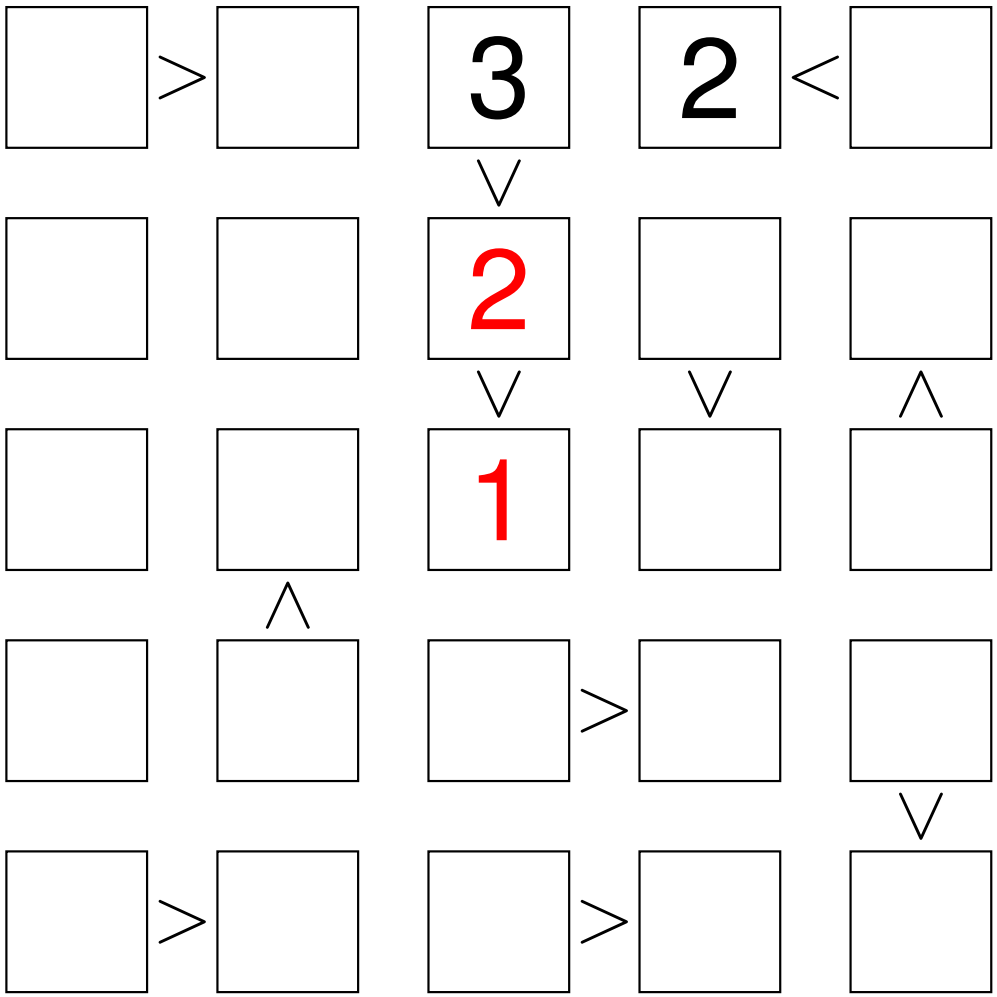
Futoshiki



Futoshiki



Futoshiki



Futoshiki

$$([d]^2, N, L, U_{1,3}, U_{1,4}) \to ([d], \neq_d, <_d, \{(3)\}, \{(2)\})$$

$$N = \{\big((i,j),(i,b)\big) \mid j \neq b\} \\ \cup \{\big((i,j),(a,j)\big) \mid i \neq a\}$$

$$L = \{\big((i,j),(a,b)\big) \mid (i,j) < (a,b)\}$$

$$U_{i,j} = \{\big((i,j)\big)\}$$

Lemma (Fellows, Hermelin, Rosamond, Vialette, 2008): kMC is W[1]-complete.

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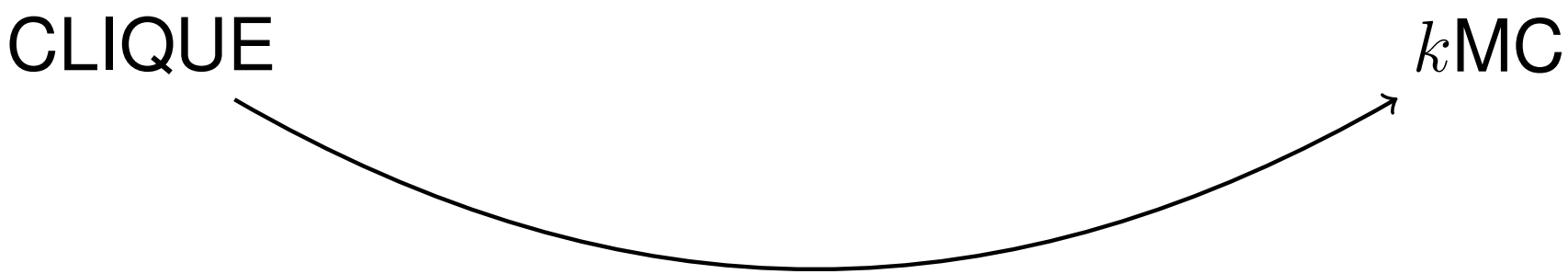
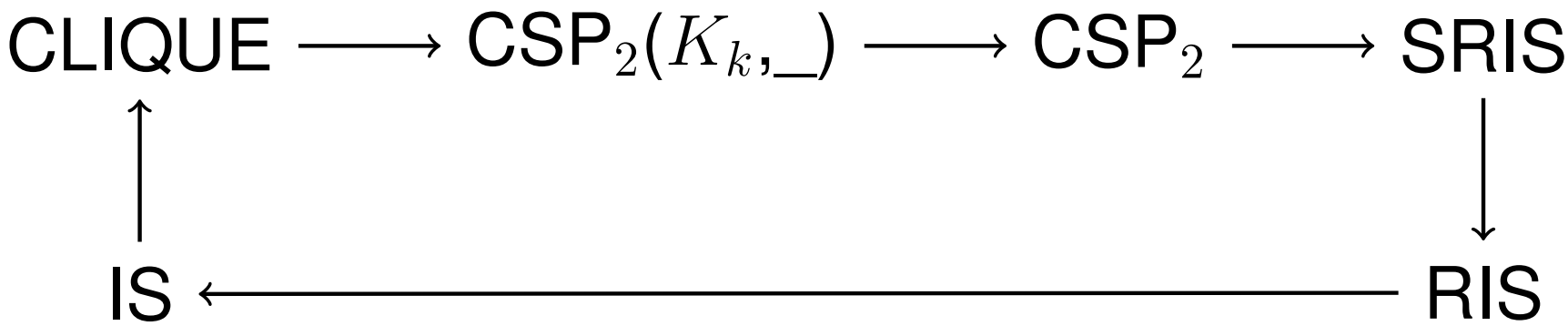
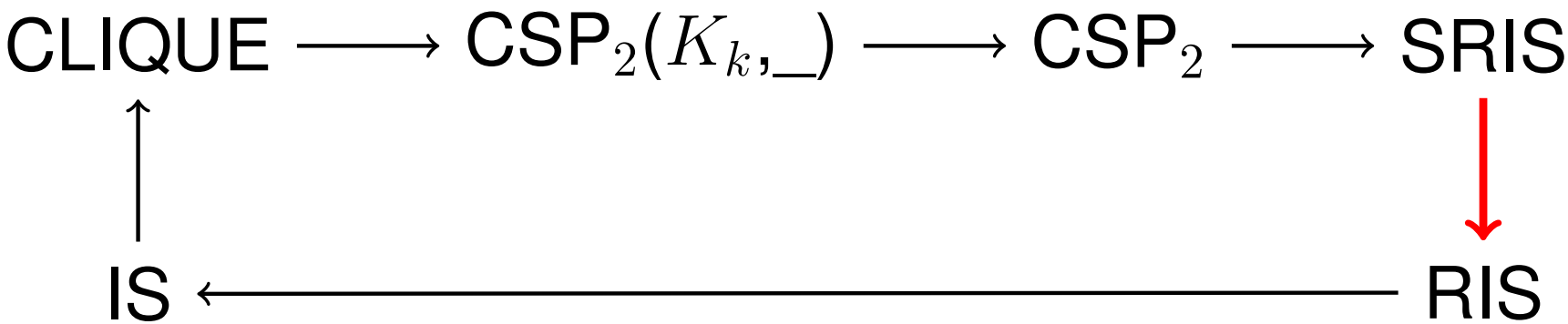


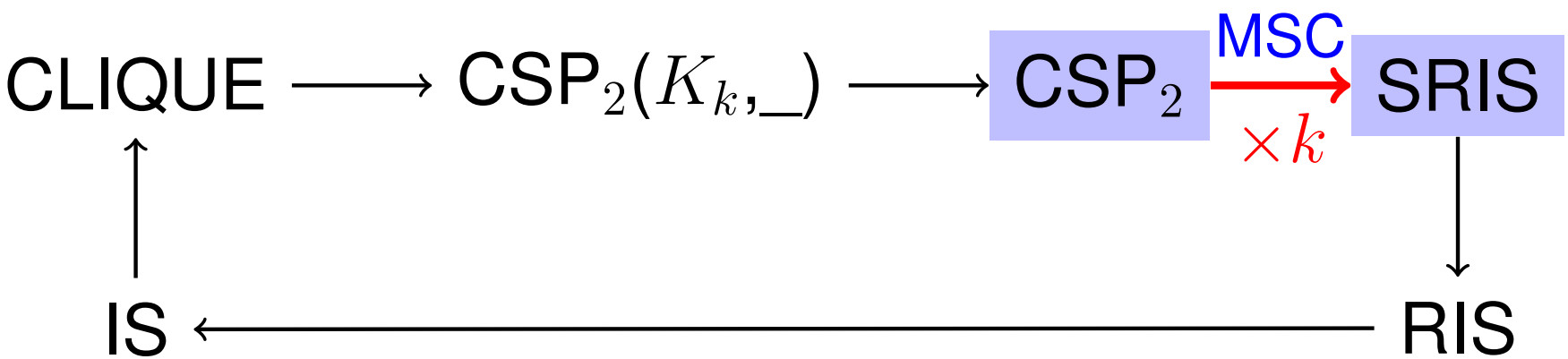
Lemma (Fellows, Hermelin, Rosamond, Vialette, 2008): kMC is W[1]-complete.

Corollary: SRIS is W[1]-complete.



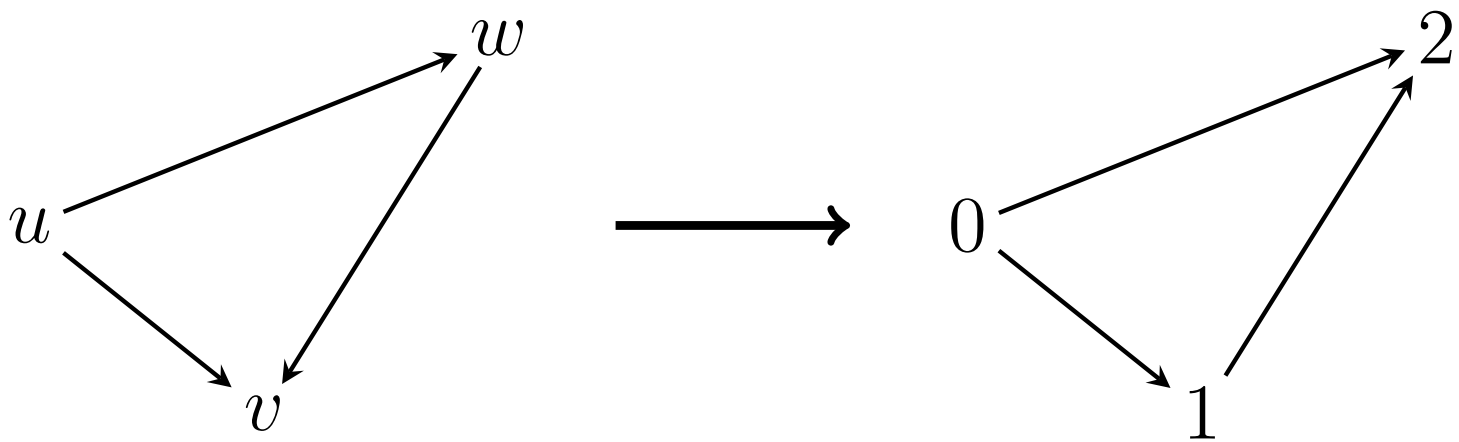
Hell 1972 (graphs), Gregory Gutin and colleagues 2006 (digraphs)


CLIQUE
CSP$_2(K_k,_)$
CSP$_2$
SRIS
IS
RIS


CLIQUE
CSP$_2(K_k,_)$
CSP$_2$
MSC
$\times k$
SRIS
IS
RIS

microstructure complement

$S = (\{u, v, w\}, \{(u, v), (u, w), (w, v)\})$
$T = (\{0, 1, 2\}, <_3)$
$S \to T$?



microstructure complement

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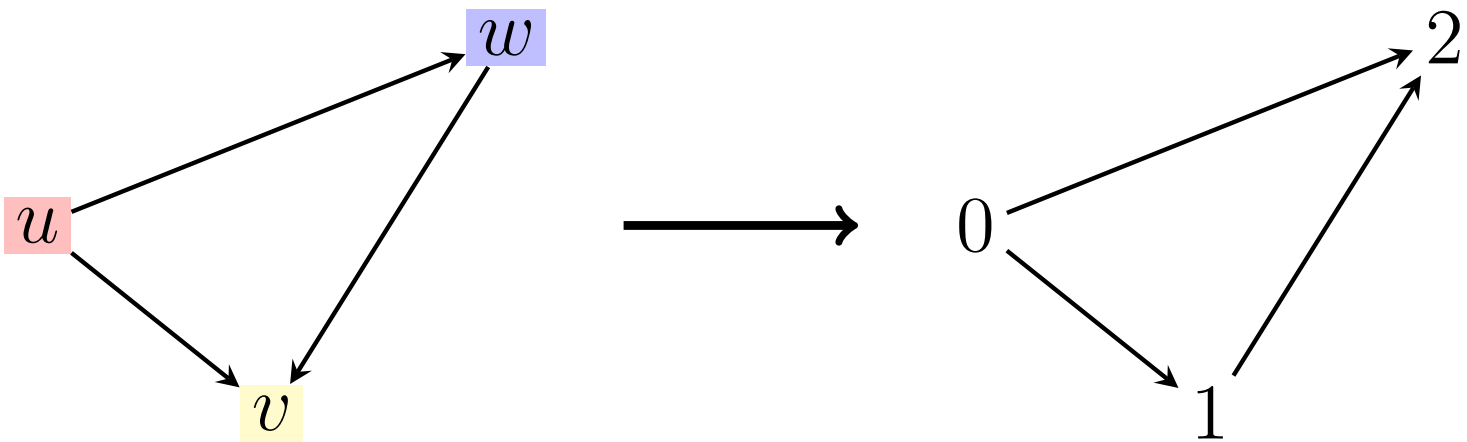
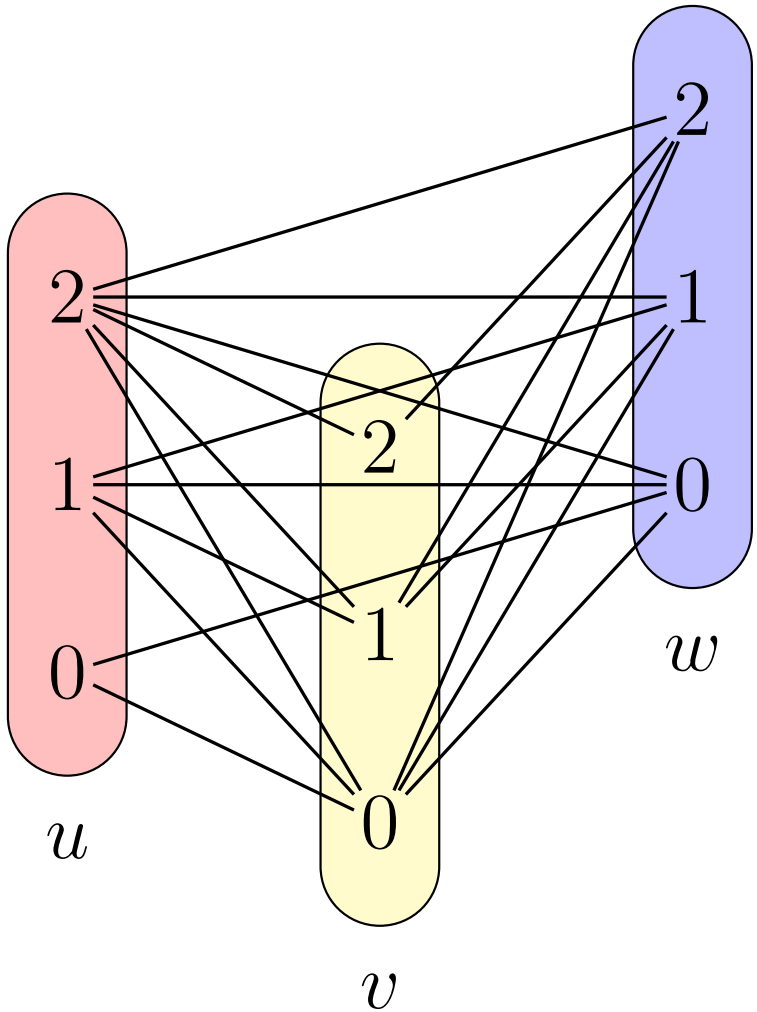


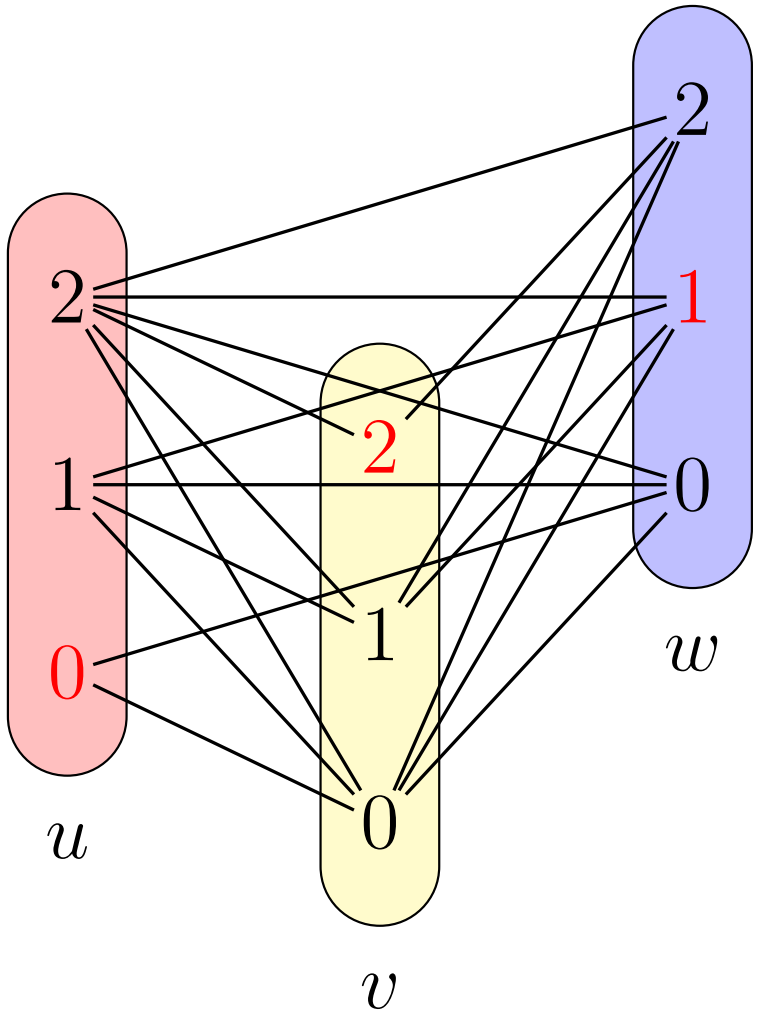
MSC

$(G(S \times \overline{T}), \pi_1)$

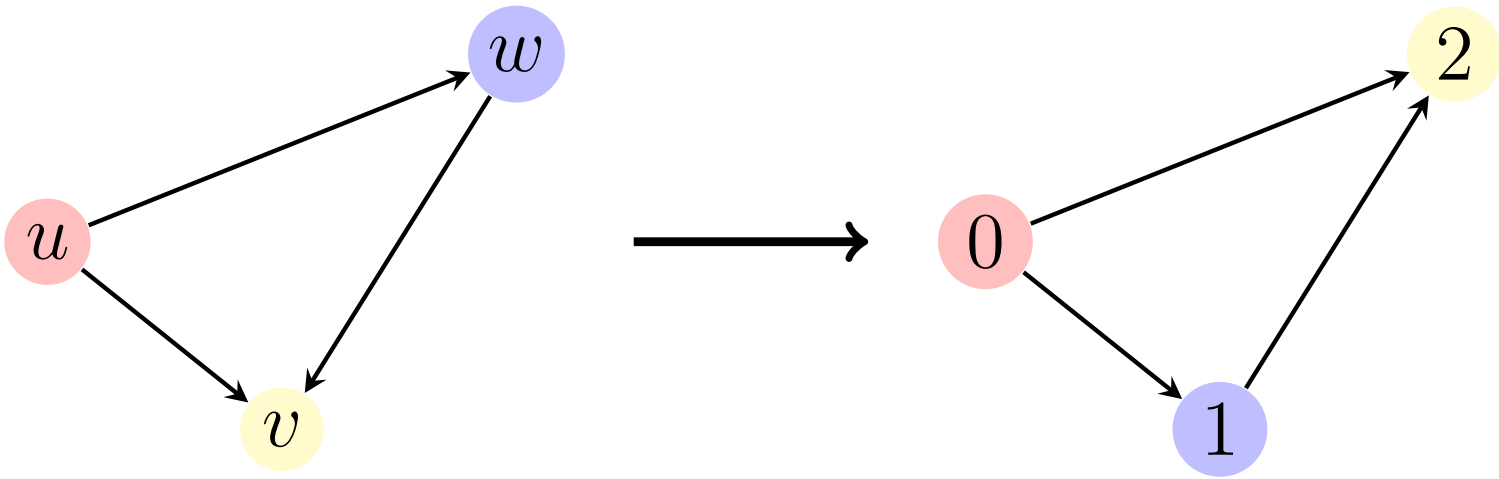


MSC

$$SRIS\big((G(S \times \overline{T}), \pi_1)\big)$$



MSC



MSC

$$(V, (Q_i)_I), (W, (R_i)_I) \qquad\qquad (S, T)$$

MSC

$$(V, (Q_i)_I), (W, (R_i)_I) \qquad (S, T)$$

$$(V \times W, (Q_i \times \overline{R_i})_I) \qquad S \times \overline{T}$$

MSC

$$(V, (Q_i)_I), (W, (R_i)_I) \qquad (S, T)$$

$$(V \times W, (Q_i \times \overline{R_i})_I) \qquad S \times \overline{T}$$

$$(V \times W, E(\bigcup_{i \in I} Q_i \times \overline{R_i})) \quad G(S \times T)$$

dragons: need to keep $\overline{R}$ small; universe must be finite

SRIS((F, c)-free)

$\downarrow$

IS(F-free) $\rightarrow$ RIS((F, c)-free)

F-free: no F induced substructure

(F, c)-free: no (F, c) induced substructure

$\forall c$

SRIS((F, c)-free) NPC

$\Downarrow$

IS(F-free) NPC $\Rightarrow$ RIS((F, c)-free) NPC

F-free: no F induced substructure

(F, c)-free: no (F, c) induced substructure

$\exists c$

SRIS((F, c)-free) PTIME

$\Uparrow$

IS(F-free) PTIME $\Leftarrow$ RIS((F, c)-free) PTIME

F-free: no F induced substructure

(F, c)-free: no (F, c) induced substructure

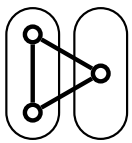
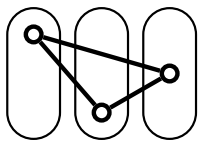
Proposition (S. and Jeavons, 2008):

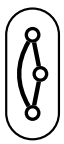
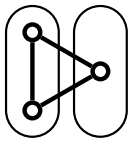
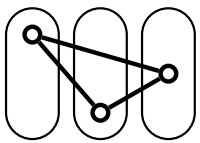
$(S, T) \in$ALLDIFF $\Rightarrow MSC(S, T) \in$PERFECT

Proposition (S. and Jeavons, 2008):

$(S, T) \in$TREE $\Rightarrow MSC(S, T) \in$PERFECT

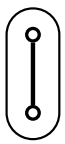
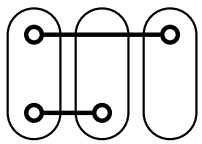
IS( -free) NPC

$\Rightarrow$ RIS(( ,  , )-free) NPC

$\Rightarrow$ $\Rightarrow$
- RIS( -free) NPC,
- RIS( -free) NPC,
- RIS( -free) NPC

Theorem (Balas and Yu, 1989): IS($2K_2$-free) PTIME

Proposition (Cooper, Jeavons, S., 2008):

SRIS( ,  -free) PTIME

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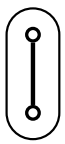
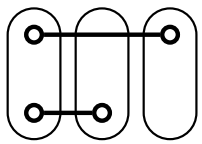
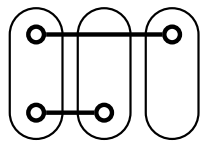
SRIS( ,  -free) PTIME

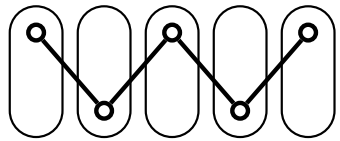
Question: RIS( -free) PTIME?

$\exists c$ RIS((P_5, c)-free) PTIME

$\Rightarrow$ IS(P_5-free) PTIME

$\exists c$ RIS((P_5, c)-free) PTIME

$\Rightarrow$ IS(P_5-free) PTIME

Question: RIS( -free) PTIME or NPC?

Further work

- Detailed analysis of all reductions
- ISGCI for vc-graphs?
- How to use colours for kernelization?
- Non-random colours for colour coding?